(*Paper No. 3042.*)

"Note on Maximum Crosshead-Velocity."

By GEORGE ARTHUR BURLS.

THE problem of determining the position of the crosshead, when its velocity is a maximum (the crank being supposed to turn uniformly), has of late attracted some attention. In Professor Hill's investigation [1] is to be found the first exact determination of this position and its value in certain selected cases. Professor Unwin shows [2] how the fundamental equation of the problem may be deduced from a geometrical construction to the crosshead acceleration, due to Rittershaus.

The ratio of connecting-rod to crank being denoted by n, the Author proposes to show how the general equation may be established from Professor Klein's graphical construction, and gives a more extensive Table of results than has hitherto appeared. Another point of interest is that in which the velocity of the crosshead is equal to that of the crank-pin; and geometrical constructions for finding this position, for values of n, both greater than and less than 1, are given. Lastly, attention is directed to a curious reciprocal property of the centrode and crosshead velocity curve.

Graphical Methods.—At least four methods are at present known of graphically determining the crosshead acceleration, viz.:—

(1) Proll's, or the "subnormal" method; (2) Rittershaus' method; (3) Mohr's method; and (4) Klein's method.

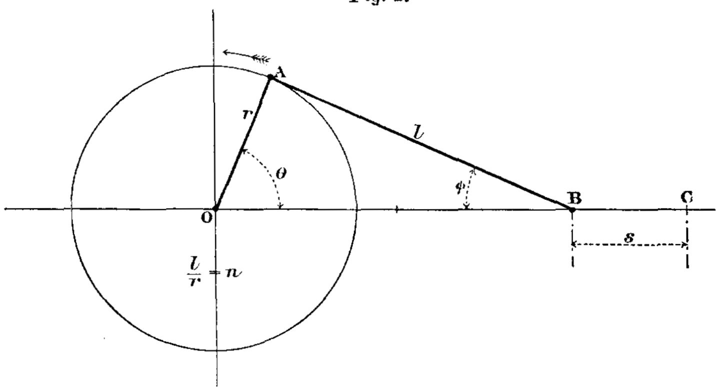
Proll's method is interesting, but of little practical value, and it fails at the dead points. Rittershaus' method also fails at the dead points, and suffers from the defects of acute intersections and requiring a large surface of paper. Mohr's method fails at the dead points, but is especially useful for values of n less than 1. Klein's method does not fail at the dead points, and is much

[1] Minutes of Proceedings Inst. C.E., vol. cxxiv. p. 390.
[2] *Ibid*, vol. cxxv. p. 363.

the best to adopt in all ordinary cases; it is inapplicable, however, to values of n less than 1.

The graphic solution of the maximum-velocity problem consists in drawing the crosshead-acceleration curve by one of the above methods, and the configuration to maximum velocity is that corresponding to zero crosshead-acceleration. There are two quantities that might be sought—the crank-angle θ, at which maximum crosshead-velocity occurs, and the corresponding distance s, passed over by the crosshead from, say, the inner dead point. No graphical procedure will give θ in any other than a

Fig. 1.



very rough way; s is, however, easily obtained graphically within $\frac{1}{4}$ per cent. of its true value. Referring to *Fig. 1*—

$$\frac{s}{r} = \text{vers}\,\theta + n\,\text{vers}\,\phi \quad . \quad . \quad . \quad . \quad (1)$$

and

$$\sin\phi = \frac{1}{n}\sin\theta \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

The analytic procedure is to differentiate equation (1) twice with respect to the time and equate the result to zero; this gives—

$$\cos\left(\theta + \phi\right) + n\left(\frac{d\,\phi}{d\,\theta}\right)^2 = 0.$$

By aid of equation (2) Professor Hill shows that this equation can be expressed in the form—

$$(n^2 - 1)\,(\sin^6\theta - n^2\sin^4\theta - n^4\sin^2\theta + n^4) = 0 \quad . \quad (3)$$

z 2

Denoting $\sin^2\theta$ by x, and $(n^2 - 1)$ being not zero, equation (3) becomes—

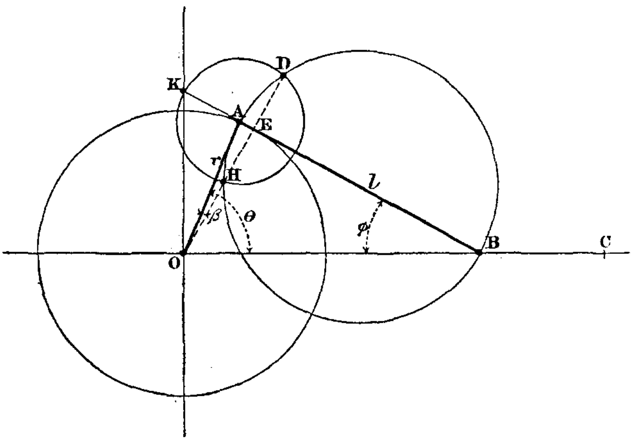
$$x^3 - n^2 x^2 - n^4 x + n^4 = 0 \quad . \quad . \quad . \quad . \quad (4)$$

This cubic in x is the fundamental equation of the problem, the required root is that lying between 0 and + 1, and the crank-angle for maximum crosshead-velocity is—

$$\theta = \sin^{-1}\sqrt{x}.$$

Deduction of General Cubic from Klein's Figure.—Klein's construction for the crosshead-acceleration is as follows: In *Fig.* 2 produce B A to K; on A B as diameter describe a circle; with A as

Fig. 2.



centre and A K as radius, cut this circle in D and H; join D H and produce it, if necessary, to cut O C. The intercept between O and the point of cutting measured the acceleration of B, to the scale on which O A measured A's velocity. Hence, for maximum velocity of B, D H produced must pass through O. No geometrical process for drawing the system in this configuration is at present known. Denoting the angle A O E by β, it follows that $\beta = (\theta + \phi) - 90$;

$$\therefore \qquad \sin\beta = -\cos(\theta + \phi) = \frac{\text{A E}}{r} = \frac{\text{A K}^2}{r\,l} \quad . \quad . \quad (5)$$

But $$\frac{\text{A K}}{r} = \frac{\sin \text{A O K}}{\sin \text{A K O}} = \frac{\cos\theta}{\cos\phi}; \quad \therefore \frac{\text{A K}^2}{r\,l} = \frac{r\cos^2\theta}{l\cos^2\phi};$$

therefore, by equation (5)—

$$\sin\theta \cdot \sin\phi - \cos\theta \cdot \cos\phi = \frac{\cos^2\theta}{n\cos^2\phi} \quad . \quad . \quad (6)$$

But by equation (2) $\sin\phi = \frac{1}{n}\sin\theta$; also $\sin^2\theta = x$; and therefore equation (6) becomes—

$$\frac{x}{n} - \sqrt{\left(1 - x\right)\left(1 - \frac{x}{n^2}\right)} = \frac{1 - x}{n\left(1 - \frac{x}{n^2}\right)} \quad . \quad (7)$$

and this equation, reduced by algebra to—

$$x^3 - n^2x^2 - n^4x + n^4 = 0 \quad . \quad . \quad . \quad . \quad . \quad (4)$$

is the general equation (4).

Solution of the General Equation (4).—It is useful to have the means of obtaining the required root of equation (4), freed from all technicalities; thus, for all values of n, from 1 to ∞ inclusive—

$$x = \frac{n^2}{3}\left\{1 + 4\sin\left(30 - \frac{\alpha}{3}\right)\right\} \quad . \quad . \quad . \quad (8)$$

where α is found from the relation—

$$\cos\alpha = \frac{1 \cdot 6875}{n^2} - 0 \cdot 6875.$$

It is by aid of this solution that the accompanying Table has been formed.

Equation (4) a quadratic in n^2.—In equation (4), again, n might be taken as the unknown and x as the known quantity; it then becomes a quadratic equation in n^2, and, proceeding in the usual manner, the solution is—

$$n^2 = \frac{x^2}{2(1 - x)}\left\{1 \pm \sqrt{5 - \frac{4}{x}}\right\} \quad . \quad . \quad . \quad (9)$$

From this result it follows that:—(1) To each value of x there correspond two values of n; (2) as $\left(5 - \frac{4}{x}\right)$ cannot be negative the least value of x is $\frac{4}{5}$; to this corresponds $n = \sqrt{\frac{8}{5}}$, and $\theta = 63° \ 26' \ 6''$, a minimum. Hence, as the maximum value of x is unity (since $x = \sin^2\theta$), if equation (9)

GENERAL TABLE OF RESULTS.

$n =$	$\frac{s}{r} =$	θ = from equation (4)	θ by Prof. Unwin's approximation.	$\sin^{-1}\frac{1}{n(n^2+2)} + \tan^{-1} n$	$\tan^{-1} n$	Error of taking $\tan^{-1} n$ for θ.
1·0	2·00000	90 0 0	..	64 28 16	45 0 0	45 0 0
1·01	1·39807	73 10 10	..	..	45 17 6	27 53 4
1·1	1·05304	64 57 50	..	..	47 43 35	17 14 15
$\frac{1+\sqrt{5}}{2\sqrt{2}}$ [1]	1·00000	64 5 11	..	..	48 50 44	15 14 27
1·2	0·95644	63 35 5	..	64 12 49	50 11 40	13 23 25
$\sqrt{\frac{8}{5}}$ [2]	0·92327	63 26 6	..	..	51 40 16	11 45 50
1·3	0·91013	63 28 1	..	..	52 25 53	11 2 8
1·4	0·88391	63 48 16	..	64 51 14	54 27 44	9 20 32
1·5	0·86809	64 20 38	..	..	56 18 36	8 2 2
$\sqrt{\frac{27}{11}}$ [3]	0·86109	64 45 38	..	..	57 27 2	7 18 36
1·6	0·85839	64 58 45	..	65 52 20	57 59 40	6 59 5
1·7	0·85253	65 39 18	..	..	59 32 4	6 7 14
1·8	0·84924	66 20 37	..	67 1 55	60 56 45	5 23 52
1·9	0·84772	67 1 45	..	..	62 14 30	4 47 15
1·974	0·847395	67 31 38	..	..	63 8 2	4 23 36
1·975	0·847393	67 32 2	..	..	63 8 44	4 23 18
1·976	0·847395	67 32 26	..	..	63 9 26	4 23 0
2·0	0·84741	67 42 0	67 43 10	68 12 55	63 26 6	4 15 54
2·25	0·84998	69 17 17	..	..	66 2 15	3 15 2
2·5	0·85504	70 43 46	..	..	68 11 55	2 31 51
2·75	0·86101	72 1 5	..	..	70 1 1	2 0 4
3·0	0·86736	73 10 31	73 10 31	73 18 5	71 33 54	1 36 37
4·0	0·89057	76 43 24	76 43 15	76 45 35	75 57 50	0 45 34
5·0	0·90845	79 6 34	79 5 58	79 6 52	78 41 24	0 25 10
6·0	0·92177	80 47 40	..	80 47 21	80 32 16	0 15 24
7·0	0·93213	82 3 3	..	82 1 50	81 52 12	0 10 51
8·0	0·93892	82 56 30	..	82 59 1	82 52 30	..
9·0	0·94684	83 47 12	..	83 44 11	83 39 35	..
10·0	0·95235	84 24 59	..	84 20 54	84 17 22	..
∞	1·00000	90 0 0	..	90 0 0	90 0 0	..

were used, only values of x, lying between 0·8 and 1·0 inclusive, are to be considered.

Evaluation of $\frac{s}{r}$.—From equations (1) and (2)—

$$\frac{s}{r} = n + 1 - \sqrt{1 - x} - \sqrt{n^2 - x} \quad . \quad . \quad (10)$$

Inserting in this the value of x obtained from equation (4) the

---

[1] $\frac{1+\sqrt{5}}{2\sqrt{2}} = 1·144123.$ [2] $\sqrt{\frac{8}{5}} = 1·2649.$ [3] $\sqrt{\frac{27}{11}} = 1·5667.$

value of $\frac{s}{r}$ is found, and values of $\frac{s}{r}$ for a series of values of n are given in the general Table. On inspection of the values of $\frac{s}{r}$ it is seen that as n increased from 1 to ∞, $\frac{s}{r}$ diminished—at first from 2 to $0 \cdot 847393$ ($n = 1 \cdot 975$), thereafter increasing to 1, so that when $n = 1 \cdot 975$ approximately $\frac{s}{r}$ is a minimum; but the determination of the exact value of n at which $\frac{s}{r}$ attained its minimum has not been effected. It is obvious, however, from the Table that $n = 1 \cdot 975$ is a close approximation to that value.

Value of n when $\frac{s}{r} = 1$.—When the Table was first constructed it was perceived that for some value of n between $1 \cdot 1$ and $1 \cdot 2$ the maximum crosshead-velocity occurred at mid-stroke, *i.e.* $\frac{s}{r} = 1$. The corresponding value of n is determined as follows:—In equation (10) put $\frac{s}{r} = 1$; then it will be found on reduction that $x = 1 - \frac{1}{4 n^2}$; substituting this value of x in equation (4), and reducing for determining n—

$$n^8 - 2\, n^6 + \frac{13}{12} n^4 - \frac{1}{4} \cdot n^2 + \frac{1}{48} = 0 \quad . \quad . \quad . \quad . \quad . \quad (12)$$

a bi-quadratic in n^2.

Solving, the required value of n has been found to be—

$$n = \frac{1 + \sqrt{5}}{2 \sqrt{2}} \, (= 1 \cdot 144123).$$

Approximate Calculation of θ.—Three methods of proceeding may be noticed:—

(1) The roughest approximation to θ is, as Professor Unwin observed, obtained by assuming that maximum crosshead-velocity occurred when the crank and connecting-rod are mutually at right-angles, so that $\tan \theta = n$; the general Table shows the extent of the error; for values of n greater than 3 it is a fairly good approximation.

(2) Professor Cotterill[1] gives an approximate expression which, for usual values of n, practically amounted to taking

$$\theta = \tan^{-1} n + \sin^{-1} \frac{1}{n\,(n^2 + 2)}.$$

Some results obtained from this formula are included in the Table.

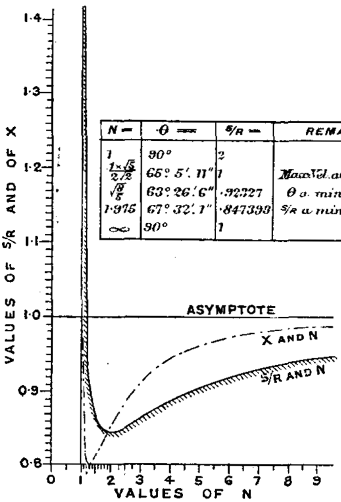
[1] "Applied Mechanics," 4th ed., p. 97.

(3) Professor Unwin's formula[1] for approximately estimating θ

$$\sin^2\theta = \frac{n^2}{n^2+1} + \frac{2\,n^2+1}{(n^2+1)(n^4+4\,n^2)}$$

gives, for values of n usual in practice, an extremely close approximation to the truth.

Fig. 3.

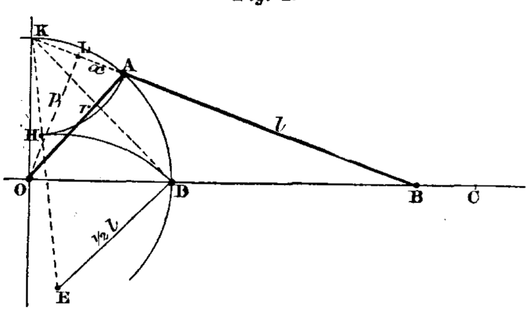


N =	θ =	S/R =	REMARKS
1	90°	2	
$\frac{1\times\sqrt{5}}{2\sqrt{2}}$	65° 5′ 11″	1	Max. Vel. at ½ Stroke
$\sqrt{\frac{8}{5}}$	63° 26′ 6″	·92327	θ a minimum
1·975	67° 32′ 1″	·847393	S/R a minimum
∞	90°	1	

In *Fig. 3* are drawn curves exhibiting the relation between $\frac{s}{r}$ and n, and between x and n.

Positions for equal Velocity of Crosshead and Crank-pin.—It is well known that four crank-positions exist for which this is the case. When n exceeds 1 two of these occur when the crank made 90° and 270° with the line of stroke. The remaining two are symmetrically situated about the line of stroke, and are determined by drawing from K, *Fig. 4*, a line K A B, such that the portion A B is equal in length to the connecting-rod. From O drop a perpendicular O L on K A; denote O L by p and L A by x. Then evidently

Fig. 4.



$$p^2 = r^2 - x^2\,;$$

and also,

$$p^2 = x\,(x + l)\,;$$

whence, $r^2 - x^2 = x\,(x + l)$; solving this quadratic in x there results

$$\mathrm{K\,A} = 2\,x = \sqrt{\left(\frac{l}{2}\right)^2 + 2\,r^2} - \left(\frac{l}{2}\right) \quad . \quad . \quad . \quad . \quad (13)$$

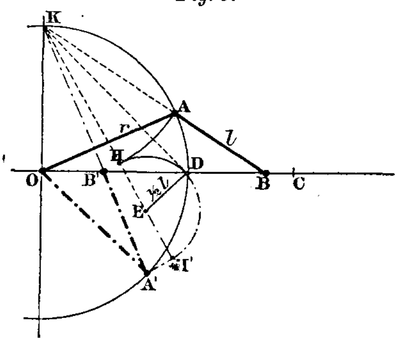
[1] Minutes of Proceedings Inst. C.E., vol. cxxv. p. 366. The misprint in the expression as originally given is corrected in the Errata, *ante*.

Now this suggested the following geometrical construction for finding A:—From D draw $DE = \left(\frac{l}{2}\right)$, at right-angles to DK, and join KE; along EK take EH = ED; and from K, as centre with radius = KH, cut the crank-pin circle. The point of cutting is the point, A, required.

Demonstration.—It is required to show that KH = KA. Now $KD^2 = 2r^2$; $DE = \left(\frac{l}{2}\right)$; $\therefore KE \sqrt{\left(\frac{l}{2}\right)^2 + 2r^2}$, and $\therefore$ by the construction, $KH = \sqrt{\left(\frac{l}{2}\right)^2 + 2r^2} - \left(\frac{l}{2}\right) = KA$.

Equal Crosshead and Crank-pin Velocities, n less than 1.—In this case the crank can only oscillate about D through an angle $2 \sin^{-1} n$.

Fig. 5.



Two cases arise, *Fig. 5*:—(1) When the crosshead is outside the crank-pin circle; and (2) when the crosshead is within the crank-pin circle. The first case is dealt with in exactly the same manner as when n was greater than 1.

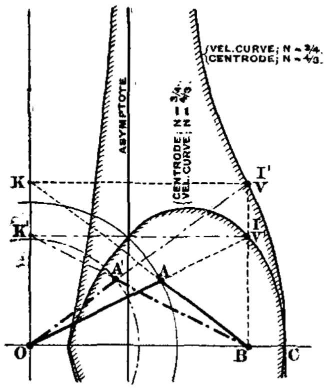
For the second case, an investigation similar to that for $n > 1$, gives as the equations for finding A':—$p^2 = r^2 - x^2$; and $p^2 = x\ (x - l)$, whence—

$$KA' = 2x = \sqrt{\left(\frac{l}{2}\right)^2 + 2r^2} + \left(\frac{l}{2}\right) \quad . \quad . \quad . \quad . \quad (14)$$

For this case, then, the construction must be modified thus:—Along KE produced, take EH' = ED; and then, from K as centre with radius KH', cut the crank-pin circle in the required point A'. It may be observed here that if n be greater than 1 the stroke is $2r$, if $n = 1$ the stroke is $2r + 2l$, and if n be less than 1 the stroke is $2l$.

Curious Reciprocal Property of Centrode and Velocity-Curve.—The well-known construction for determining points on the crosshead velocity-curve is as follows:—Produce B A to meet a perpendicular, through O, to the line of stroke in K; then OK measured the crosshead-velocity to the scale on which O A measured that of the crank-pin. And BV = OK; also, the instantaneous centre of the connecting-rod is obtained by producing O A to meet a perpendicular at B in I; and the locus of I is the centrode of the connecting-rod. The familiar velocity-curve and centrode are immediately recognized in *Fig. 6.*

Fig. 6.



Now let the crank and connecting-rod be interchanged, then the velocity-curve and centrode are interchanged; thus when n $(= \frac{4}{3}$ in *Fig.*) becomes $\frac{1}{n}$, the inner finite curve is the centrode and the curve with the two infinite branches is the crosshead-velocity curve. This is apparent from the complete symmetry of the rectilinear construction-figure within the rectangle B O K V.

The Paper is accompanied by two drawings, from which the *Figs.* in the text have been prepared.